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STUDY THE SPECTRUM OF THE UPPER TRIANGULAR MATRIX $U(p, 0, q, 0, r)$ OVER THE SEQUENCE SPACES c_0 AND c

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Abstract. In this paper, an analysis has been made on the spectrum and fine spectrum of the upper triangular matrix $U(p, 0, q, 0, r)$ over the sequence spaces c_0 and c. We also investigate the approximate point spectrum and compression spectrum on these spaces.

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1. Introduction

Operator theory is a notable branch of mathematics due to its immense application in diverse section of physical world. The study of Matrices is fundamental model for operator theory. It has been observed that the spectral theory contributes to a great extent in extension of the eigenvalues theory for matrices. Owing to a huge number of appliances of Spectral theory in different field of science, researchers are now motivating towards this arena of mathematics.

A lots of research has been made on summability methods related to spectrum and fine spectrum of matrix operators on some sequence spaces. But there is still a lot to be examined on spectra of some matrix operators transforming one class of sequences into another class of sequences.

Akhmedov and Basar [1], Altay and Basar [2–4] made analysis on the spectra of difference operator $\triangle$ and generalized difference operator on c_0, c, ℓ_p and bv_p under various conditions. In recent period, the fine spectrum of $B(r, s, t)$ over the sequence spaces c_0 and c; and ℓ_p and bv_p has been examined by Furkan et al. [6, 7]. Tripathy and Paul [12, 13] have considered the case of the spectra and fine spectra of the operator $D(r, 0, 0, s)$ and $D(r, 0, s, 0, t)$ over the sequence spaces c_0 and c respectively. Further, Tripathy and Paul [15] have studied the spectrum and fine spectrum of the operator $D(r, 0, s, 0, t)$ over the sequence spaces ℓ_p and bv_p.The fine spectra of the upper triangular matrix $A(r, s, t)$ as well as lower triangular matrix $B(r, s, t)$ on the sequence spaces c and ℓ_p, where $(0 < p < 1)$ have been explored by Karaisa et al. [9]. Srivastava and Kumar [16] have observed the spectra and the fine spectra of the generalized difference operator $\triangle_v$ on ℓ_1.Further, a lots of research works on spectra has been carried out by Okutoyi [11] and Rath and Tripathy [12] and others.

Throughout the paper we denote $w, \ell_\infty, c, c_0, \ell_p$ and bv_p be the space of all, bounded, convergent, null, p-absolutely summable and bounded variation sequences respectively.

2. Preliminaries and Definitions

Let Y be a linear space. We denote $B(Y)$ be the set of all bounded linear operators on Y into itself. If $L \in (Y)$, where Y is a Banach space then the adjoint operator L^* of L is a bounded linear operator on the dual L^* of L defined by $(L^*\varnothing)(y) = \varnothing(Ly)$ for all $\varnothing \in L^*$ and $y \in Y$.

Let $L : D(L) \longrightarrow Y$ be a linear operator, defined on $D(L) \subset Y$, where $D(L)$ denote the domain of L and Y is a complex normed linear space. For $L \in B(Y)$ we associate a complex number β with the operator $(L - \beta I)$ denoted by L_β defined on the same domain $D(L)$, where I is the identity operator. The inverse $(L-\beta I)^{-1}$, denoted by L_β^{-1} is known as the resolvent operator of L_β.

A complex number β is called a regular value of L if it satisfies the following conditions
$(B_1)L_\beta^{-1}$ exists,
$(B_2)L_\beta^{-1}$ is bounded and,
$(B_3)L_\beta^{-1}$ is defined on a set which is dense in Y.

The collection of all regular values β of L is called the resolvent set of L and is denoted by $\rho(L, Y)$. The complement of the resolvent set over the set of complex $\mathbb{C}$ is called the spectrum of L and is denoted by $\sigma(L, Y)$. Thus the spectrum $\sigma(L, Y)$ consist of $\beta \in \mathbb{C}$, for which L_β is not invertible.

Classification of spectrum:

The spectrum $\sigma(L, Y)$ is partitioned into three disjoint sets, which are as follows:
(i) The point (discrete) spectrum denoted by $\sigma_p(L, Y)$ consist of $\beta \in \mathbb{C}$, for which L_β^{-1} does not exist. The elements of point spectrum are called the eigen values of L.
(ii) The continuous spectrum denoted by $\sigma_c(L, Y)$ consist of $\beta \in \mathbb{C}$, for which L_β^{-1} exists and satisfies (B_3) but not (B_2) that is L_β^{-1} is unbounded.
(iii) The residual spectrum denoted by $\sigma_r(L, Y)$ consist of $\beta \in \mathbb{C}$, for which L_β^{-1} exists (and may be bounded or not) but not satisfy (B_3), that is, the domain of L_β^{-1} is not dense in Y.

This is to be noted that for finite dimensional case, $\sigma_c(L, Y) = \sigma_r(L, Y) = \varnothing$ and hence $\sigma_p(L, Y) = \sigma(L, Y)$.

Appell et al. [5], has been given more classification of spectrum, which are mentioned below:

Given a bounded linear operator L in a Banach space Y, we call a sequence (y_k) in Y as a Weyl sequence for L if $||y_k|| = 1$ and $||Ly_k|| \longrightarrow 0$, as $k \longrightarrow \infty$.

(a)The approximate point spectrum: $\sigma_{ap}(L, Y) = \{\beta \in \mathbb{C} : \textit{there exist a Weyle sequence for } L_\beta\}$

(b)The compression spectrum: $\sigma_{co}(L, Y) = \{\beta \in \mathbb{C} := \overline{R(L_\beta)} \neq Y\}$

Proposition 2.1 [[5], Proposition 1.3, p.28] *Spectra and subspectra of an operator $L \in B(Y)$ and its adjoint $L^* \in B(Y^*)$are related by the following relations*: $(i)\sigma(L^*, Y^*) = \sigma(L, Y)$,
$(ii)\sigma(L^*, Y^*) \subseteq \sigma_{ap}(L, Y)$,
$(iii)\sigma_p(L^*, Y^*) = \sigma_{co}(L, Y)$,
$(iv)\sigma_{co}(L^*, Y^*) \supseteq \sigma_p(L, Y)$,

$(v)\sigma(L,Y) = \sigma_{ap}(L,Y) \cup \sigma_p(L^*,Y^*)\sigma_{co}(L^*,Y^*) \supseteq \sigma_p(L,Y) = \sigma_p(L,Y) \cup \sigma_{ap}(L^*,Y^*)$,

Let E and F be two sequence spaces and $B = (b_{nk})$ be an infinite matrix of real or complex numbers b_{nk}, where $n, k \in \mathbb{N} = \{1, 2, \ldots\}$. Then,it can be stated that B defines a matrix mapping from E into F, denote by $B : E \to F$, if for every sequence $y = (y_n) \in E$, the sequence $By = \{(By)_n\}$ is in F where $(Bx)_n = \sum_{k=1}^{\infty} b_{nk}\, y_k$ provided the right hand side converges for every $n \in \mathbb{N}$ and $y \in E$. By $(E : F)$ we represent the class of all matrices B such that $B : E \longrightarrow F$.

3. classification of spectrum of $U(p,0,q,0,r)$ over the space of convergent sequences

Let r, s, t be non-zero real numbers, and we define the upper triangular matrix as follows

$$U(p,0,q,0,r) = \begin{pmatrix} p & 0 & q & 0 & s & 0 & \cdots \\ 0 & p & 0 & q & 0 & s & \cdots \\ 0 & 0 & p & 0 & q & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \end{pmatrix}$$

Lemma 3.1. *The matrix $B = (b_{nk})$ gives rise to a bounded linear operator $L \in B(c)$ from c to itself if and only if*
(i) *the rows of B in ℓ_1 and their ℓ_1 norms are bounded,*
(ii) *the columns of B are in c,*
(iii) *the sequence of row sums of B is in c.*
The operator norm of L is the supremum of the ℓ_1 norms of the rows.

For the above Lemma one may refer to Theorem 1.3.6 of Goldberg [8].

Corollary 3.1: *$U(p,0,q,0,r) : c \longrightarrow c$ is a bounded linear operator with* $||U(p,0,q,0,r)||_{(c,c)} = |p| + |q| + |r|$.

Lemma 3.2 [see [8], p. 60] *L has a dense range if and only if L^* is one to one, where L^* denote the adjoint operator of L.*

If $L : c \longrightarrow c$ is a bounded matrix operator with matrix B, then $L^* : c^* \longrightarrow c^*$ acting on $c \oplus \ell_1$ has a matrix depiction of the structure $\begin{pmatrix} \chi & 0 \\ b & B^t \end{pmatrix}$ where χ is the limit of the sequence of row sums of B minus the sum of the columns of B, and b is the column vector whose kth entry is the limit of the kth column of B for each $k \in \mathbb{N}$. For $U(p,0,q,0,r) : c \longrightarrow c$, the matrix $U(p,0,q,0,r)^* \in B(\ell_1)$ is of the form $U(p,0,q,0,r)^* = \begin{pmatrix} p+q+r & 0 \\ 0 & U(p,0,q,0,r)^t \end{pmatrix}$

Theorem 3.3 *$U(p,0,q,0,r) : c \longrightarrow c$ has a dense range if and only if $\beta \neq p+q+r$.*

Proof. To prove the result it is sufficient to show that $\sigma_p[U(p,0,q,0,r)^*, \mathbb{C} \oplus \ell_1] = p+q+r$.
Let, β be an eigen value of the operator $U(p,0,q,0,r)^* : \mathbb{C} \oplus \ell_1 \longrightarrow \mathbb{C} \oplus \ell_1$. Then there exist a non-zero vector $v \in \ell_1$ satisfy the following system of equations

$$(p+q+r)v_1 = \beta v_1$$

$$pv_2 = \beta v_2$$

$$qv_2 + pv_4 = \beta v_4$$

$rv_2 + qv_4 + pv_6 = \beta v_6$

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$$\tag{3.1}$$

$pv_3 = \beta v_3$

$qv_3 + pv_5 = \beta v_5$

$rv_3 + qv_5 + pv_7 = \beta v_7$

- - -

$$\tag{3.2}$$

Now, it is clear from the above system of equations $\beta = p + q + r$ is an eigen value corresponds to the eigen vector $(1, 0, 0, - - -)$. Next consider $\beta \neq p + q + r$ then $v_1 = 0$. Let, v_m be the first non-zero entry of the sequence $v = (v_n)$ then from the above system of equations (3.1) and (3.2) we have $rv_{m-4} + qv_{m-2} + pv_m = \beta v_m$ we obtain that $p = \beta$ and since $q \neq 0$ from the next equation of either (3.1) or (3.2) we get $v_m = 0$ which is a contradiction and hence $p + q + r$ is the only eigen value of the given matrix. This completes the proof.

□

Theorem 3.4: $\sigma_{co}[U(p, 0, q, 0, r), c] = p + q + r$.

Proof. The result follows from the Theorem 3.3 and the proposition 2.1. □

Theorem 3.5:*Let q be a complex number such that $\sqrt{q^2} = -q$ and define the set*
$L_1 = \{\gamma \in \mathbb{C} : |-q + \sqrt{q^2 - 4r(p - \gamma)}| \geq 2|p - \gamma|\}$
Then $\sigma_c[U(p, 0, q, 0, r), c] \subseteq L_1$.

Proof. Let $v = (v_k) \in \ell_1$. Then, by solving the equation $[U(p, 0, q, 0, r)^* - \beta I]u = v$ for $u = (u_k)$ in terms of v, we have,

$u_1 = \frac{v_1}{p+q+r-\beta}$,

$u_2 = \frac{v_2}{p-\beta}$,

$u_3 = \frac{v_3}{p-\beta}$,

$u_4 = \frac{v_4}{p-\beta} + \frac{-qv_2}{(p-\beta)^2}$,

$u_5 = \frac{v_5}{p-\beta} + \frac{-qv_3}{(p-\beta)^2}$,

$u_6 = \frac{v_6}{p-\beta} + \frac{-qv_4}{(p-\beta)^2} + \frac{\{q^2-r(p-\beta)\}v_2}{(p-\beta)^3}$,

$u_7 = \frac{v_7}{p-\beta} + \frac{-qv_5}{(p-\beta)^2} + \frac{\{q^2-r(p-\beta)\}v_3}{(p-\beta)^3}$,

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Consider,$a_1 = \frac{1}{p-\beta}$ and $a_2 = \frac{-q}{(p-\beta)^2}$ then we obtain by recursively that $a_n = -\frac{qa_{n-1}+ra_{n-2}}{p-\beta}$ for $n \geq 3$
Now, the above equations can be rewritten as follows

$u_1 = \frac{v_1}{p+q+r-\beta}$,

$u_2 = a_1 v_2$,

$u_3 = a_1 v_3$,

$u_4 = a_1 v_4 + a_2 v_2$,

$u_5 = a_1 v_5 + a_2 v_3$,

$u_6 = a_1 v_6 + a_2 v_4 + a_3 v_2$,

$u_7 = a_1 v_7 + a_2 v_5 + a_3 v_3$,

$u_8 = a_1 v_8 + a_2 v_6 + a_3 v_4 + a_4 v_2$,

$u_9 = a_1 v_9 + a_2 v_7 + a_3 v_5 + a_4 v_3$,

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$u_n = a_1 v_n + a_2 v_{n-2} + a_3 v_{n-4} + - - - + a_{\frac{n}{2}} v_2$, when n is even

$u_n = a_1 v_n + a_2 v_{n-2} + a_3 v_{n-4} + - - - + a_{\frac{(n-1)}{2}} v_3$, when n is odd

We have,

$a_n = -\frac{q a_{n-1} + r a_{n-2}}{p-\beta}$

$\Rightarrow (p-\beta) a_n + q a_{n-1} + r a_{n-2} = 0$

The characteristic equation of the recurrence relation is $(p-\beta)\mu^2 + q\mu + r = 0$.
If $\Delta = q^2 - 4(p-\beta)r \neq 0$, then one can easily straightforward calculate that,

$$a_n = \frac{\mu_1^n - \mu_2^n}{\sqrt{q^2 - 4r(p-\beta)}}, \forall n \geq 1, \mu_1 = \frac{-q+\sqrt{\Delta}}{2(p-\beta)}, \mu_2 = \frac{-q-\sqrt{\Delta}}{2(p-\beta)} \tag{3.3}$$

Now, We have,

$|u_n| \leq |a_1||v_n| + |a_2||v_{n-2}| + |a_3||v_{n-4}| + - - - + |a_{[\frac{n}{2}]}||v_2|$, when n is even and

$|u_n| \leq |a_1||v_n| + |a_2||v_{n-2}| + |a_3||v_{n-4}| + - - - + |a_{[\frac{n}{2}]}||v_3|$, when n is odd

where [ ], denote greatest integer function.
Let, n be even then,

$$(|u_2|+|u_4|+|u_6|+---|u_n|) \leq (|a_1|+|a_2|+|a_3|+---+|a_{[\frac{n}{2}]}|)|v_2|+(|a_1|+|a_2|+|a_3|+---+|a_{[\frac{n}{2}]-1}|)|v_4|+(|a_1|+|a_2|+|a_3|+---+|a_{[\frac{n}{2}]-2}|)|v_6|+---+(|a_1|+|a_2|)|v_{n-2}|+|a_1||v_n|$$

$$\leq (|a_1|+|a_2|+|a_3|+---+|a_{[\frac{n}{2}]}|)(|v_2|+|v_4|+|v_6|+---+|v_n|) \tag{3.4}$$

Similarly, when n is odd we have,

$$(|u_3|+|u_5|+|u_7|+---|u_n|) \leq (|a_1|+|a_2|+|a_3|+---+|a_{[\frac{n}{2}]}|)|v_3|+(|a_1|+|a_2|+|a_3|+---+|a_{[\frac{n}{2}]-1}|)|v_5|+(|a_1|+|a_2|+|a_3|+---+|a_{[\frac{n}{2}]-2}|)|v_7|+---+(|a_1|+|a_2|)|v_{n-2}|+|a_1||v_n|$$

$$\leq (|a_1|+|a_2|+|a_3|+---+|a_{[\frac{n}{2}]}|)(|v_3|+|v_5|+|v_7|+---+|v_n|)$$

$$(3.5)$$

Adding (3.4) and (3.5) with $|u_1|$ we have, for all $n \in \mathbb{N}$

$(|u_1|+|u_2|+|u_3|+|u_4|+---|u_n|) \leq \frac{|v_1|}{|p+q+r-\beta|}+(|a_1|+|a_2|+|a_3|+---+|a_{[\frac{n}{2}]}|)(|v_2|+|v_4|+|v_6|+---+|v_n|)+(|a_1|+|a_2|+|a_3|+---+|a_{[\frac{n}{2}]}|)(|v_3|+|v_5|+|v_7|+---+|v_n|)$,

$=\frac{|v_1|}{|p+q+r-\beta|}+(|a_1|+|a_2|+|a_3|+---+|a_{[\frac{n}{2}]}|)(|v_1|+|v_2|+|v_3|+|V_4|+---+|v_n|)$

By letting $n \longrightarrow \infty$, we have,

$||u||_1 \leq \frac{|v_1|}{|p+q+r-\beta|}+||v||_1 \sum_{k=1}^{\infty}|a_k|$

Our aim is to show

$\sum_{k=1}^{\infty}|a_k| < \infty$

Two cases arise

Case 1 : Let, $\Delta = q^2-4(p-\beta)r \neq 0$ then relation (3.3) holds for all $n \in \mathbb{N}$.

First we prove that if $|\mu_1| < 1$ then $|\mu_2| < 1$.

Let, $|\mu_1| < 1$, then $|-q+\sqrt{q^2-4r(p-\beta)}| < 2|p-\beta|$

Since, $\sqrt{q^2} = -q$ we have, $|1+\sqrt{1-\frac{4r(p-\beta)}{q^2}}| < |\frac{2(p-\beta)}{-q}|$

. Again, since $|1-\sqrt{q}| \leq |1+\sqrt{q}|$ for any $q \in \mathbb{C}$,

We have,$|1-\sqrt{1-\frac{4r(p-\beta)}{q^2}}| < |\frac{2(p-\beta)}{-q}|$ which implies that $|\mu_2| < 1$.

Now, since $\sum_{k=1}^{\infty}|a_k| \leq \frac{1}{|\sqrt{\Delta}|}(\sum_{k=1}^{\infty}|\mu_1|^k+\sum_{k=1}^{\infty}|\mu_2|^k)$, hence for $|\mu_1| < 1$ we arrive that $(u_k) \in \ell_1$.This implies that $U(p,0,q,0,r)^*-\beta I$ is onto. Thus, by the Lemma 3.2, we can conclude that $U(p,0,q,0,r)-\beta I$ has a bounded inverse and hence $\sigma_c[U(p,0,q,0,r),c] \subseteq L_1$.

Case 2 : Let, $\Delta = q^2-4(p-\beta)r = 0$ consider $\sqrt{\Delta} = \delta$

From the recurrence relation, we have

$a_n = \frac{\mu_1^n-\mu_2^n}{\sqrt{q^2-4r(p-\beta)}}$

$=\frac{1}{\{2(p-\beta)\}^n\delta}\{(-s+\delta)^n-(-s-\delta)^n\}$

$=\frac{1}{\{2(p-\beta)\}^n\delta}\{2n(-s)^{n-1}\delta+2C(n,3)(-s)^{n-3}\delta^3+---\}$, where, $C(n,r) = \frac{n!}{r!(n-r)!}$

$=\frac{1}{\{2(p-\beta)\}^n}\{2n(-s)^{n-1}+2C(n,3)(-s)^{n-3}\delta^2+---\}$

Now, if $\Delta = 0$ implies $\delta = 0$ and we have,

$$a_n = (\frac{2n}{-q})\{\frac{-q}{2(p-\beta)}\}^n, \forall n \in \mathbb{N} \tag{3.6}$$

Again, for $|-q| < 2|p-\beta|$ we can see that

$lim_{n \longrightarrow \infty}|\frac{a_n}{a_{n-1}}| = |\frac{-q}{2(p-\beta)}| < 1$

Therefore, $\sum_{k=1}^{\infty}|a_k| < \infty$ and hence $(u_k) \in \ell_1$. Hence, by the same argument as in the case 1, we can conclude that $\sigma_c[U(p,0,q,0,r),c] \subseteq L_1$. □

Theorem 3.6: *Define the set* L_2 *by* $L_2 = \{\gamma \in \mathbb{C} : |-q+\sqrt{q^2-4r(p-\gamma)}| > 2|p-\gamma|\}$, *then* $\sigma_p[U(p,0,q,0,r),c] = L_2$

Proof. Let, $U(p,0,q,0,r)u = \beta u$ for $u \neq (0,0,0,---) \in c$.
Then by solving the system of linear equations,

$$pu_1 + qu_3 + ru_5 = \beta u_1$$

$$pu_2 + qu_4 + ru_6 = \beta u_2$$

$$pu_3 + qu_5 + ru_7 = \beta u_3$$

$$pu_4 + qu_6 + ru_8 = \beta u_4$$

- - -

$$pu_{k-4} + qu_{k-2} + ru_k = \beta u_{k-4}, \ \forall, k \geq 5$$

- - -

We have

$$u_5 = \tfrac{-q}{r}u_3 - \tfrac{p-\beta}{r}u_1$$

$$u_6 = \tfrac{-q}{r}u_4 - \tfrac{p-\beta}{r}u_2$$

$$u_7 = \tfrac{q^2-r(p-\beta)}{r^2}u_3 + \tfrac{q(p-\beta)}{r^2}u_1$$

$$u_8 = \tfrac{q^2-r(p-\beta)}{r^2}u_4 + \tfrac{q(p-\beta)}{r^2}u_2$$

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If $p = \beta$ then we may choose $u_1 \neq 0$ then $u = (u_1, 0, 0, ---)$ is an eigen vector corresponding to $p = \beta$ and hence $\beta \in \sigma_p[U(p,0,q,0,r), c]$. Next, let $p \neq \beta$, then from the above equations we have,for all $n \geq 3$

$$u_{2n+1} = \frac{a_n(p-\beta)^n}{r^{n-1}}u_3 - \frac{a_{n-1}(p-\beta)^n}{r^{n-1}}u_1 \quad (3.7)$$

$$u_{2n+2} = \frac{a_n(p-\beta)^n}{r^{n-1}}u_4 - \frac{a_{n-1}(p-\beta)^n}{r^{n-1}}u_2 \quad (3.8)$$

Assume that $\beta \in L_2$. Then we choose $u_1 = u_2 = 1$ and $u_3 = \frac{2(p-\beta)}{-q+\sqrt{q^2-4r(p-\beta)}}$ and $u_4 = \frac{u_3}{2}$
Since μ_1 and μ_2 are the roots of the characteristic equation $(p-\beta)\mu^2 + q\mu + r = 0$, we must have $\mu_1\mu_2 = \frac{r}{p-\beta}$ and $\mu_1 - \mu_2 = \frac{\sqrt{\Delta}}{p-\beta}$
Combining $u_3 = \frac{1}{\mu_1}$ with relation (3.7) we have

$$u_{2n+1} = \tfrac{a_n(p-\beta)^n}{r^{n-1}}u_3 - \tfrac{a_{n-1}(p-\beta)^n}{r^{n-1}}u_1$$

$$= (\tfrac{(p-\beta)}{r})^{n-1}(p-\beta)(a_n u_3 - a_{n-1}u_1)$$

$$(3.9)$$

$$= \tfrac{1}{(\mu_1\mu_2)^{n-1}}\tfrac{p-\beta}{\sqrt{\Delta}}(-\mu_1^{n-1} + \mu_2^{n-1} + \mu_1^{n-1} - \mu_2^n\mu_1^{-1})$$

$$= \tfrac{1}{\mu_1^{n-1}\mu_2^{n-1}}(\tfrac{1}{\mu_1-\mu_2})\mu_2^{n-1}(\tfrac{\mu_1-\mu_2}{\mu_1})$$

$$= \tfrac{1}{\mu_1^n}$$

$= (u_3)^n$

Similarly we can show that, $u_{2n+2} = (u_4)^n$.
Next let, $\Delta = 0$ then $q^2 - 4(p-\beta)r = 0$ and hence $p - \beta = \frac{q^2}{4r}$ and $\mu_1 = \mu_2 = \frac{-2r}{q}$ therefore from equation (3.6) we get,

$a_n = (\frac{2n}{-q})(\frac{-2r}{q})^n$

Then, substituting $u_1 = u_2 = 1, u_3 = u_4 = \frac{-q}{2r}$ and a_n in equation (3.9) we get,

$u_{2n+1} = (\frac{q}{-2r})^{2n-2}\frac{q^2}{4r}\{\frac{2n-2}{q}(\frac{-2r}{q})^{n-1} - \frac{2n}{q}(\frac{-2r}{q})(\frac{q}{-2r})\}$

$= (\frac{q}{-2r})^{2n-2}(\frac{-2r}{q})^{n-1}\frac{q^2}{4r}(\frac{2n-2}{q} - \frac{2n}{q})$

$= (\frac{q}{-2r})^n(\frac{-2r}{q})\frac{q^2}{4r}(\frac{-2}{q})$

$= (\frac{q}{-2r})^n$

$= (u_3)^n$

Similarly we can show that, $u_{2n+2} = (u_4)^n$ in this case also.
Now,$lim_{n\longrightarrow\infty}\frac{u_{2n+2}}{u_{2n+1}} = lim_{n\longrightarrow\infty}(\frac{u_4}{u_3})^n = lim_{n\longrightarrow\infty}(\frac{1}{2})^n = 0 < \infty$,therefore $u = (u_n) \in c$ and hence $\beta \in \sigma_p[U(p,0,q,0,r),c]$ i.e.$L_2 \subseteq \sigma_p[U(p,0,q,0,r),c]$.
Next, we assume that $\beta \notin L_2$ and to show that $\beta \notin \sigma_p[U(p,0,q,0,r),c]$.
Since $\beta \notin L_2$ implies that $|\mu_1| \leq 1$. We study the following three cases:

Case 1 : Let, $|\mu_2| < |\mu_1| < 1$ then we have $q^2 - 4(p-\beta)r \neq 0$ and from the relation (3.7) we get,

$u_{2n+1} = \frac{a_n(p-\beta)^n}{r^{n-1}}u_3 - \frac{a_{n-1}(p-\beta)^n}{r^{n-1}}u_1$

$= (\frac{p-\beta}{r})^{n-1}(p-\beta)(-a_{n-1}u_1 + a_n u_3)$

$= \frac{p-\beta}{\sqrt{\Delta}(\mu_1\mu_2)^{n-1}}(-\mu_1^{n-1}u_1 + \mu_2^{n-1}u_1 + \mu_1^{n-1}u_3 - \mu_2^n u_3)$

$= \frac{p-\beta}{\sqrt{\Delta}}\{(\frac{1}{\mu_1^{n-1}} - \frac{1}{\mu_2^{n-1}})u_1 + (\frac{\mu_1}{\mu_2^{n-1}} - \frac{\mu_2}{\mu_1^{n-1}})u_3\}$

$= \frac{p-\beta}{\sqrt{\Delta}}\{\frac{1}{\mu_1^{n-1}}(u_1 - \mu_2 u_3) + \frac{1}{\mu_2^{n-1}}(-u_1 + \mu_1 u_3)\}$

If we consider $u_1 - \mu_2 u_3 = 0$ and $u_1 + \mu_1 u_3 = 0$ then $\mu_1 = \mu_2$, a contradiction and hence $\beta \notin \sigma_p[U(p,0,q,0,r),c]$.

Case 2 : Let, $|\mu_2| = |\mu_1| < 1$. then $q^2 - 4(p-\beta)r = 0$ and from the relation (3.6) we have,

$$a_n = (\frac{2n}{-q})\{\frac{-q}{2(p-\beta)}\}^n, \forall n \in \mathbb{N} \tag{3.10}$$

Substituting (3.10) into (3.9), we get the following

$u_{2n+1} = \frac{2(p-\beta)}{q\mu_1^{n-1}}\{u_1(n-1) - nu_3\mu_1\}$

Similarly we have,

$u_{2n+2} = \frac{2(p-\beta)}{q\mu_2^{n-1}}\{u_1(n-1) - nu_4\mu_1\}$

Let, $u_1 = u_2 = u_3 = u_4 = 0$, the we have, $u = \theta$, a contradiction and hence $\beta \notin \sigma_p[U(p,0,q,0,r),c]$.

Case 3 : Let, $|\mu_1| = |\mu_2| = 1$ then we have $q^2 - 4(p-\beta)r = 0$ and so $|\frac{-q}{2r}| = 1$. Substitute (3.10) into (3.9) we have,

$$u_{2n+1} = (\tfrac{-q}{2r})^{n-1}\{-(n-1)(\tfrac{-q}{2r})u_1 + nu_3\}$$

Similarly we have,

$$u_{2n+2} = (\tfrac{-q}{2r})^{n-1}\{-(n-1)(\tfrac{-q}{2r})u_2 + nu_4\}$$

If $u_1 = u_2 = u_3 = u_4 = 0$, we get $u = \theta$, a contradiction and hence $\beta \notin \sigma_p[U(p,0,q,0,r),c]$.
Thus in all the cases we have $\sigma_p[U(p,0,q,0,r),c] \subseteq L_2$. This completes the theorem. □

Theorem 3.7:$\sigma_p[U(p,0,q,0,r)^*, c^*] = \{p+q+r\}$.

Proof. We have, $\sigma_p[U(p,0,q,0,r)^*, c^*] = \sigma_{co}[U(p,0,q,0,r),c] = \{p+q+r\}$. □

Theorem 3.8: $\sigma_r[U(p,0,q,0,r),c] = \{p+q+r\}$.

Proof. The result follows from the theorem $\sigma_r[U(p,0,q,0,r),c] = \sigma_{co}[U(p,0,q,0,r),c] \setminus \sigma_p[U(p,0,q,0,r),c]$. □

Theorem 3.9: $\sigma[U(p,0,q,0,r),c] = L_1$, *where* L_1 *is define as in Theorem*3.5.

Proof. Since, $\sigma_p[U(p,0,q,0,r),c] \subseteq \sigma[U(p,0,q,0,r),c]$, therefore,

$$\{\gamma \in \mathbb{C} : |-q + \sqrt{q^2 - 4r(p-\gamma)}| > 2|p-\gamma|\} \subseteq \sigma[U(p,0,q,0,r),c]$$

Since the spectrum of any bounded operator is closed [10], we have,

$$\{\gamma \in \mathbb{C} : |-q + \sqrt{q^2 - 4r(p-\gamma)}| \geq 2|p-\gamma|\} \subseteq \sigma[U(p,0,q,0,r),c]$$

Again, from Theorem 3.5, Theorem 3.6 and Theorem 3.8 we can conclude that

$$\sigma[U(p,0,q,0,r),c] \subseteq \{\gamma \in \mathbb{C} : |-q + \sqrt{q^2 - 4r(p-\gamma)}| \geq 2|p-\gamma|\}$$

Thus,

$$\sigma[U(p,0,q,0,r),c] = \{\gamma \in \mathbb{C} : |-q + \sqrt{q^2 - 4r(p-\gamma)}| \geq 2|p-\gamma|\} = L_1$$

□

Theorem 3.9: $\sigma_c[U(p,0,q,0,r),c] = L_3 \setminus \{p+q+r\}$, *where*
$L_3 = \{\gamma \in \mathbb{C} : |-q + \sqrt{q^2 - 4r(p-\gamma)}| = 2|p-\gamma|\}$

Proof. Since, $\sigma[U(p,0,q,0,r),c]$ is the disjoint union of $\sigma_p[U(p,0,q,0,r),c]$, $\sigma_r[U(p,0,q,0,r),c]$ and $\sigma_c[U(p,0,q,0,r),c]$ therefore result follows from Theorem 3.6, Theorem 3.8 and Theorem 3.9. □

4. **classification of spectrum of $U(p,0,q,0,r)$ over the space of null sequences**

Lemma 4.1. *The matrix* $B = (b_{nk})$ *gives raise to a bounded linear operator* $L \in B(c_0)$ *from* c_0 *to itself if and only if*

(1) *the rows of* B *in* ℓ_1 *and* ℓ_1 *their norms are bounded,*
(2) *the columns of* B *are in* c_0.

The operator norm of L *is the supremum of the* ℓ_1 *norms of the rows.*

For the above Lemma one may refer to Example 8.4.5A of Goldberg [8].

Corollary 4.1: *$U(p,0,q,0,r): c_0 \longrightarrow c_0$ is a bounded linear operator with $||U(p,0,q,0,r)||_{(c,c)} = ||U(p,0,q,0,r)||_{(c_0,c_0)}$.*

If $L: c_0 \longrightarrow c_0$ is a bounded linear operator with the matrix B, then it is known that the adjoint operator $L^*: c_0^* \longrightarrow c_0^*$ is defined by the transpose B^t of the matrix B. It should be noted that the dual space c_0^* of c_0 is isometically isomorphic to the Banach space ℓ_1 of absolutely summable sequences normed by $||y|| = \sum |y_n|$.

Theorem 4.2: $\sigma_p[U(p,0,q,0,r)^*, c_0^*] = \varnothing$.

Proof. Suppose $U(p,0,q,0,r)^*u = \beta u$ for $u \neq \theta = (0,0,0,---) \in c_0$. Then by solving the system of linear equations we have,

$$\begin{aligned} &pu_1 = \beta u_1 \\ &qu_1 + pu_3 = \beta u_3 \\ &ru_1 + qu_3 + pu_5 = \beta u_5 \\ &- - - \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &pu_2 = \beta u_2 \\ &qu_2 + pu_4 = \beta u_4 \\ &ru_2 + qu_4 + pu_6 = \beta u_6 \\ &- - - \end{aligned} \tag{4.2}$$

If u_m is the first non-zero entry of the sequence $u = (u_m)$ then from the previous system of linear equations (4.1) and (4.2), we have $ru_{m-4} + qu_{m-2} + pu_m = \beta u_m$ and we obtain that $\beta = p$ and from the next of either (4.1) or (4.2) we get $u_m = 0$ which is a contradiction .This completes the proof. □

Theorem 4.3: $\sigma_{co}[U(p,0,q,0,r), c_0] = \varnothing$.

Proof. We have $\sigma_{co}[U(p,0,q,0,r), c_0] = \sigma_p[U(p,0,q,0,r)^*, c_0^*] = \varnothing$, follows from the Theorem 4.2. □

Theorem 4.4: $\sigma_r[U(p,0,q,0,r), c_0] = \varnothing$.

Proof. One has $\sigma_p[U(p,0,q,0,r)^*, c_0^*] = \varnothing$, therefore $U(p,0,q,0,r)^* - \beta I$ is not one- to- one for all $\beta \in \mathbb{C}$ and by Lemma 3.2, we can conclude that $U(p,0,q,0,r)^* - \beta I$ have a dense range for all $\beta \in \mathbb{C}$ and consequently, $\sigma_r[U(p,0,q,0,r), c_0] = \varnothing$. □

Theorem 4.5: $\sigma_p[U(p,0,q,0,r), c_0] = L_2$, *where L_2 is defined as in Theorem* 3.6.

Proof. This is obtained in the similar way used in the proof of Theorem 3.6. □

Theorem 4.6: $\sigma_c[U(p,0,q,0,r),c_0] = L_1$, *where* L_1 *is defined as in Theorem* 3.5.

Proof. This is obtained in the similar way used in the proof of Theorem 3.5. □

Theorem 4.7: $\sigma[U(p,0,q,0,r),c_0] = L_1$, *where* L_1 *is defined as in Theorem* 3.5.

Proof. This is obtained in the similar way used in the proof of Theorem 3.9. □

Theorem 4.8:$\sigma_c[U(p,0,q,0,r),c_0] = L_3$, *where* L_3 *is define as in Theorem* 3.10.

Proof. Since, $\sigma[U(p,0,q,0,r),c_0]$ is the disjoint union of $\sigma_p[U(p,0,q,0,r),c_0]$, $\sigma_r[U(p,0,q,0,r),c_0]$ and $\sigma_c[U(p,0,q,0,r),c_0]$ therefore result follows from Theorem 4.4, Theorem 4.5 and Theorem 4.7. □

Theorem 4.9: $\sigma_{ap}[U(p,0,q,0,r),c_0] = L_3$, *where* L_3 *is define as in Theorem* 3.10.

Proof. Since, $\sigma[U(p,0,q,0,r),c_0] = \sigma_{ap}[U(p,0,q,0,r),c_0] \cup \sigma_{co}[U(p,0,q,0,r),c_0]$, therefore by using Theorem 4.3, we can conclude $\sigma[U(p,0,q,0,r),c_0] = \sigma_{ap}[U(p,0,q,0,r),c_0]$,
since $\sigma_{co}[U(p,0,q,0,r),c_0] = \varnothing$ and hence, $\sigma_{ap}[U(p,0,q,0,r),c_0] = L_3$.

□

References

[1] A.M. Akhmedov and F.Basar, *The fine spectra of the difference operator* Δ *over the sequence space* bv_p, $(1 \le p < \infty)$, Acta Math. Sin. (Eng. Ser.) **23(10)** (2007), 1757–1768.

[2] B.Altay and F.Basar, *On the fine spectra of the difference operator* Δ *on* c_0 and c. Inform. Sci. **168** (2004), 217224.

[3] B.Altay and F.Basar, *On the fine spectra of the generalized difference operator* $B(r,s)$ *over the sequence space* c_0 and c, Int.J. Math. Sci. **18**(2005), 3005–3013.

[4] B.Altay and F.Basar, *The fine spectrum of the matrix domain of the difference operator* Δ *on the sequence space* $\ell_p(0 < p < 1)$. Commun. Math. Anal. **2(2)**(2007), 1-11.

[5] J.Appell, E.De Pascale and A.Vignoli, *Nonlinear spectral theory*, Walter de Gruyter, Berlin, New York, **10**(2008).

[6] H.Furkan, H.Bilgiç and B.Altay, *On the fine spectrum of the operator* $B(r,s,t)$ *over* c_0 *and* c. Comput.Math. Appl. **53(6)**(2007), 989-998.

[7] H.Furkan, H.Bilgiç and F.Basar, *On the fine spectrum of the operator* $B(r,s,t)$ *over the sequence spaces* ℓ_p *and* $bv_p(1 \le p < \infty)$. Comput. Math. Appl. **60**(2010), 2141-2152.

[8] S.Goldberg, *Unbounded linear operators: Theory and applications*, Courier Corporation. 2006.

[9] A.Karaisa, Y.Asar and D.T.Tollu, *Fine spectra of triangular triple-band matrices on sequence spaces c and* $\ell_p(0 < p < 1)$, Math. Commun. **21**(2016), 65-83.

[10] E.Kreyszig, *Introductory functional analysis with applications*, New York: wiley, **1**(1978).

[11] I.J.Okutoyi, *On the spectrum of* C_1 *as an operator on* bv_0, J. Austr. Math. Soc. Ser. A. **48**(1990), 79-86.

[12] D. Rath and B.C.Tripathy, *A note on spectra of operator Schur matrices and the maximal group of the algebra of triangular conservative matrices*, Indian J. Pure Appl. Math. **23(6)** (1992), 411-417.

[13] B.C.Tripathy and A.Paul, *The spectrum of the operator* $D(r,0,0,s)$ *over the sequence spaces* c_0 *and* c. Kyungpook Math. J. **53(2)**(2013), 247-256.

[14] B.C.Tripathy and A.Paul, *The spectrum of the operator* $D(r,0,s,0,t)$ *over the sequence spaces* c_0 *and* c, Jour. Math.(2013) Artical ID 430965, 7 pages.

[15] B.C.Tripathy and A.Paul, *The spectrum of the operator* $D(r,0,s,0,t)$ *over the sequence spaces* ℓ_p *and* bv_p. Afrika Matematika. **26(5-6)**(2015), 1137–1151.

[16] P.D. Srivastava and S.Kumar, *Fine spectrum of the generalized difference operator* Δ_v *on sequence space* ℓ_1, Thai J. Math. **8**(2010), 7-19.

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